# $U(1)$ gauge theory over discrete space-time and phase transitions

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## Abstract

We first apply Connes' noncommutative geometry to a finite point space. The explicit form of the action functional of $U(1)$ gauge field on this $n$-point space is obtained. We then consider the case when the $n$-point space is replaced by {space-time} $\times$ {$n$-point space}. This action is shown to relate the Hamiltonian of the continuous-spin formulation of the Potts model. We argue that $U(1)$ gauge theory on the discrete space-time determines the geometric origin of a class of phase transitions.

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## 1. Introduction

Within the framework of Connes' noncommutative geometry [1, 2], the Higgs field and the symmetry breaking mechanism in the standard model have a remarkable geometrical picture. The Higgs field is a connection, which arises from the geometry of the two-point space [3, 4]. (See also [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15], and the references therein.) This suggests that the discrete geometry may play an important role in physics. Differrential calculus and gauge theory on discrete groups was proposed in [16]. The systematic approach towards the differential calculus and the gauge theory on arbitrary finite or countable sets was formulated in [17, 18]. It is natural to ask: what is the explicit form of the action functional of gauge fields on the $n$-point space? Here $n$ is the natural number ($n \geq 2$). When $n = 2$, the action is just the case of Higgs field. The purpose of the present paper is to answer this question for the simplest case, i.e., the gauge group is taken to be $U(1)$.

In Section 2 we will review differential calculus on $n$-point space which is formulated by Dimakis and Müller-Hoissen [17, 18], and then in Section 3 we will construct the spectral triple on $n$-point space and derive the explicit form of action functional of $U(1)$ gauge fields. In section 4 the action funcional over $\{\text{space-time}\} \times \{n\text{-point space}\}$ is obtained. In the concluding section, we comment on the physical meaning of the action functional we derived and its relation to the theory of phase transitions.

## 2. Differential calculus on $n$-point space

In this section we shall review the differential calculus on $n$-point space. More detailed account of the construction can be found in [17, 18].

Let $M$ be a set of $n$ points $i_1, \cdots, i_n$ ($n < \infty$), and $\mathcal{A}$ the algebra of complex functions on $M$ with $(fg)(i) = f(i)g(i)$. Let $p_i \in \mathcal{A}$ defined by

$$p_i(j) = \delta_{ij}. \tag{1}$$

Then it follows that

$$p_i p_j = \delta_{ij} p_j, \qquad \sum_i p_i = 1. \tag{2}$$

In other words, $p_i$ is a projective operator in $\mathcal{A}$. Each $f \in \mathcal{A}$ can be written as

$$f = \sum_i f(i) p_i, \tag{3}$$

where $f(i) \in \mathbf{C}$, a complex number. The algebra $\mathcal{A}$ can be extended to a universal differential algebra $\Omega(\mathcal{A})$. The differentials satisfy the following relations:

$$p_i dp_j = -(dp_i)p_j + \delta_{ij} dp_i, \tag{4}$$

$$\sum_i dp_i = 0. \tag{5}$$

This means that the differential calculus over $n$-point space $M$ associates with it $n-1$ linear independent differentials. There is a natural geometrical representation associated with $M$. Let the projective operators $p_i$ $(i = 1, \cdots, n)$ be the orthonormal base vectors in the Euclidean space $\mathbf{R}^n$. Then $M$ forms the vertices of the $(n-1)$-dimensional hypertetrahedron embeded in $\mathbf{R}^n$.

The universal first order differential calculus $\Omega^1$ is generated by $p_i dp_j (i \neq j)$, $i, j = 1, 2, \cdots, n$. Notice that $p_i dp_i$ is the linear combinations of $p_i dp_j (i \neq j)$.

Similarly, the compositions of $p_i dp_j (i \neq j)$, $i, j = 1, 2, \cdots, n$ generate the higher order universal differential calculus on $M$. For example, the universal second order differential calculus $\Omega^2$ is generated by $p_i dp_j p_j dp_k (i \neq j, j \neq k)$, $i, j, k = 1, 2, \cdots, n$.

A simple calculation shows that

$$dp_i = \sum_j (p_j dp_i - p_i dp_j). \tag{6}$$

Furthermore,

$$dp_i dp_j = \sum_k (p_k dp_i p_i dp_j - p_i dp_k p_k dp_j + p_i dp_j p_j dp_k). \tag{7}$$

Any 1-form $\alpha$ can be written as $\alpha = \sum_{i,j} \alpha_{ij} p_i dp_j$ with $\alpha_{ij} \in \mathbf{C}$ and $\alpha_{ii} = 0$. One then finds

$$d\alpha = \sum_{i,j,k} (\alpha_{jk} - \alpha_{ik} + \alpha_{ij}) p_i dp_j p_j dp_k. \tag{8}$$

Now we consider a connection $\alpha$ on $M$, $\alpha$ is a 1-form, and skew -adjoint, $\alpha^* = -\alpha$. $\alpha$ obeys the usual transformation rule,

$$\alpha' = u\alpha u^* + u du^*. \tag{9}$$

Here $u = \sum_i u(i) p_i \in \mathcal{A}$, and $u(i) \in U(1)$, the Abelian unitary group. $\alpha$ is thus called the $U(1)$ gauge field on $M$. In order to make the formulae concise, we introduce

$$a = \sum_{i,j} (1 + \alpha_{ij}) p_i dp_j. \tag{10}$$

Notice that

$$a_{ii} = 1. \tag{11}$$

One then has

$$\begin{aligned} a' &= u a u^*, \\ a'_{ij} &= u(i) a_{ij} u(j)^*. \end{aligned} \tag{12}$$

The curvature of the connection $\alpha$ is given by

$$\theta = d\alpha + \alpha^2, \tag{13}$$

and transforms in the usual way, $\theta' = u\theta u^*$. As a 2-form, $\theta$ can be written as

$$\begin{aligned} \theta &= \sum_{i,j,k} \theta_{ijk} p_i dp_j p_j dp_k, \\ \theta_{ijk} &= a_{ij}a_{jk} - a_{ik}. \end{aligned} \tag{14}$$

**3. From spectral triple to action functional over $M$**

In noncommutative geometry, all the geometrical data is determined by a spectral triple $(\mathcal{A}, \mathcal{H}, D)$, where $\mathcal{A}$ is an involutive algebra, $\mathcal{H}$ is a Hilbert space with an involutive representation $\pi$ of $\mathcal{A}$, $D$ is a self-adjoint operator acting on $\mathcal{H}$.

We now construct the spectral triple $(\mathcal{A}, \mathcal{H}, D)$ over the $n$-point space $M$. In our case, $\mathcal{A}$ is the algebra on $M$ defined in the last section. Without loss of generality, $\mathcal{H}$ is taken to be the $n$-dimensional linear space over $\mathbf{C}$, i.e., $\mathcal{H}$ is just the direct sum $\mathcal{H} = \bigoplus_{i=1}^n \mathcal{H}_i$, $\mathcal{H}_i = \mathbf{C}$. The action of $\mathcal{A}$ is given by

$$f \in \mathcal{A} \longrightarrow \begin{pmatrix} f(1) & 0 & ... & 0 \\ 0 & f(2) & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & f(n) \end{pmatrix}. \tag{15}$$

Then $D$ is the Hermitian $n \times n$ matrix with elements $D_{ij} = D^*_{ji}$, and $D_{ij}$ is a linear mapping from $\mathcal{H}_j$ to $\mathcal{H}_i$. We have the following equality defines the involutive representation of $da$ $(a \in \mathcal{A})$ in $\mathcal{H}$,

$$\pi(da) = i[D, \pi(a)]. \tag{16}$$

To ensure the differential $d$ satisfies

$$d^2 = 0, \tag{17}$$

one has to impose the following condition on $D$,

$$D^2 = \mu^2 I, \tag{18}$$

where $\mu$ is a real constant and $I$ is the $n \times n$ unit matrix.

We now take $D_{ij} \neq 0$ $(i \neq j)$. Then the representation $\pi : \ \Omega(\mathcal{A}) \rightarrow \mathcal{L}(\mathcal{H})$ is injective on $\Omega^1(\mathcal{A})$.

In Connes' language [2], Our spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is odd (except the case of $n = 2$). For the sake of convenience, we omit the symbol $\pi$ from now on.

The projective operator $p_i$ can be expressed as the $n \times n$ matrix now,

$$(p_i)_{\alpha\beta} = \delta_{\alpha i}\delta_{\beta i}. \tag{19}$$

Notice that $D$ commute exactly with the action of $\mathcal{A}$, for the sake of convenience, we can ignore the diagonal elements of $D$, i. e.,

$$D_{ii} = 0. \tag{20}$$

From (16) and (19), one has

$$(p_i dp_j)_{\alpha\beta} = i\delta_{\alpha i}\delta_{\beta j} D_{ij}, \tag{21}$$

$$(p_i dp_j p_j dp_k p_k dp_l p_l dp_r)_{\alpha\beta} = \delta_{\alpha i}\delta_{\beta r} D_{ij} D_{jk} D_{kl} D_{lr}. \tag{22}$$

One can define an inner product $<|>$ in $\Omega^*\mathcal{A}$ by setting

$$< \alpha|\beta >= tr_\omega(\alpha^* \beta |D|^{-p}). \tag{23}$$

where $tr_\omega$ is the Dixmier trace. In our case $p = 0$ and $tr_\omega$ is reduced to the ordinary trace,

$$< \alpha|\beta >= tr(\alpha^* \beta). \tag{24}$$

Since the gauge field $\alpha$ satisfies $\alpha^* = -\alpha$, one has $\theta^* = \theta$. Then the action functional of $\theta$ is

$$S = \|\theta\|^2 =< \theta|\theta >= tr\theta^2. \tag{25}$$

From (14), (22) and (25), we have

$$S = \sum_{i,j,k,l} \theta_{ijk}\theta_{kli} D_{ij} D_{jk} D_{kl} D_{li}. \tag{26}$$

Denote

$$a_{ij} D_{ij} = H_{ij}, \tag{27}$$

where $a_{ij}$ is defined in (10). Then $H = (H_{ij})$ is a Hermitian matrix with

$$H_{ii} = 0. \tag{28}$$

From (14), (18), (26), (27) and (28), one thus has

$$S = trH^4 - 2\mu^2 trH^2 + n\mu^4. \tag{29}$$

From (28), the eigenvalues $\lambda_i$ $(i = 1, 2, \cdots, n)$ of $H$ satisfy:

$$\sum_{i=1}^{n} \lambda_i = 0. \tag{30}$$

(29) can be written as the following:

$$S = \sum_{i=1}^{n} \lambda_i^4 - 2\mu^2 \sum_{i=1}^{n} \lambda_i^2 + n\mu^4. \tag{31}$$

The eigenvalues of such kind of $H$'s generate $(n-1)$-dimensional Euclidean space $\mathbf{R}^{n-1}$. By the quadric transformation, (31) can be taken the form as

$$S = C_2^{ijkl} \sum_{i,j,k,l=1}^{n-1} \varphi_i\varphi_j\varphi_k\varphi_l - C_1 \sum_{i=1}^{n-1} \varphi_i^2 + n\mu^4, \tag{32}$$

where $(\varphi_1, \cdots, \varphi_{n-1})$ is a vector in $\mathbf{R}^{n-1}$, $C_2^{ijkl}$ and $C_1$ are real constants.

For the sake of convenience, we identify $\mathbf{R}^{n-1}$ with a subspace embedded in the $n$-dimensional geometrical representation space of $M$ introduced in Section 2 from now on. In the $(n-1)$-dimensional rectangular coordinate system, the reference point is taken to be the center of the $(n-1)$-dimensional hypertetrahedron. $M$ can then be represented by a set of $n$ vectors in $\mathbf{R}^{n-1}$: $e_i^\alpha$ $(\alpha = 1, \cdots, n;\ i = 1, \cdots, n-1)$, such that

$$\sum_i e_i^\alpha e_i^\beta = \frac{n}{n-1}\delta^{\alpha\beta} - \frac{1}{n-1}. \tag{33}$$

In (33) we have chosen the normalization of the vectors to be unity for convenience. This set of $e$'s satisfy

$$\sum_\alpha e_i^\alpha \;=\; 0; \tag{34}$$

$$\sum_\alpha e_i^\alpha e_j^\alpha \;=\; \frac{n}{n-1}\delta_{ij}. \tag{35}$$

It should be mentioned that the properties of $M$ is encoded in those of the set of spin states in the Potts model [19]. The reason will be discussed in section 5. Using (34), the eigenvalues of $H$ is

$$\lambda_\alpha = \sum_{i=1}^{n-1} \phi_i e_i^\alpha, \qquad \alpha = 1, \cdots, n. \tag{36}$$

Here $\phi_i (i = 1, \cdots, n-1)$ is a real parameter. We call $\mathbf{\Phi} = (\phi_1, \cdots, \phi_{n-1})$ the order parameter field in $\mathbf{R}^{n-1}$. Finally from (29), (35) and (36), we obtain the explicit form of $S$ over the $n$-point space $M$:

$$S = \sum_{i,j,k,l} (\sum_\alpha e_i^\alpha e_j^\alpha e_k^\alpha e_l^\alpha) \phi_i \phi_j \phi_k \phi_l - \frac{2n}{n-1}\mu^2 (\sum_i \phi_i^2) + n\mu^4. \tag{37}$$

Notice that the constant term $n\mu^4$ can be left out.

When $n = 2$, (37) changes into

$$S = 2(\phi^2 - \mu^2)^2. \tag{38}$$

This is just the Hamiltonian density of the Landau phenomenological theory of phase transitions below the critical temperature [20]. Here $\phi$ is known as the order parameter. Notice that the size of the coefficients of $\phi^2$ and $\phi^4$ does not affect the values of critical exponents of phase transitions, but it may affect the mass value of the Higgs field when (38) is considered as the Higgs potential: From (18), (27) and (36), we have $\phi^2 = \mu^2 |a_{12}|^2$. One then has

$$S = 2\mu^4 (|a_{12}|^2 - 1)^2, \tag{39}$$

which is the form of Connes' version of Higgs potential.

**4. Action functional over $V \times M$**

Now we construct the $U(1)$ gauge field theory over {space-time} $\times$ {$n$-point space}. Denote space-time by $V$, $\mathcal{A}$ the algebra of complex functions on $V \times M$. Just as in Section 2, Each $f \in \mathcal{A}$ can be written as

$$f = \sum_i f(i) p_i. \tag{40}$$

Notice that this time $f(i)$ is a complex function over $V_i$, the $ith$ copy of $V$. $\mathcal{A}$ can be also extended to a universal differential algebra $\Omega(\mathcal{A})$. Denote the differential on $M$ by $d_f$. In other words, The differential $d$ in Section 2 and Section 3 is replaced by $d_f$. Let $d_s$ be the differential on $V$, and $d$ the total differential on $V \times M$. One then has

$$d = d_s + d_f. \tag{41}$$

The nilpotency of $d$ requires that

$$d_s d_f = -d_f d_s. \tag{42}$$

Differentiate (40), we have

$$df = \sum_i (d_s f(i)) p_i + \sum_i f(i) d_f p_i. \tag{43}$$

Any 1-form $\alpha$ can be written as

$$\alpha = \sum_{i,j} \alpha_{ij} p_i d_f p_j + \sum_i \alpha_i p_i, \tag{44}$$

with $\alpha_{ij}$, the complex function on $V$ and $\alpha_{ii} = 0$; $\alpha_i$, the 1-form on $V_i$. One then finds

$$\begin{aligned} d\alpha &= \sum_{ij} (d_s \alpha_{ij}) p_i d_f p_j + \sum_{i,j,k} (\alpha_{jk} - \alpha_{ik} + \alpha_{ij}) p_i d_f p_j p_j d_f p_k \\ &\quad + \sum_i (d_s \alpha_i) p_i - \sum_i \alpha_i d_f p_i. \end{aligned} \tag{45}$$

Now we consider a connection $\alpha$ on $V \times M$, $\alpha$ is a 1-form and skew-adjoint, i.e., $\alpha$ is given by (44) and $\alpha^* = -\alpha$. $\alpha$ obeys the usual transformation rule,

$$\alpha' = u \alpha u^* + u d u^*. \tag{46}$$

Here $u = \sum_i u(i) p_i \in \mathcal{A}$, and $u(i) \in U(1)$, the Abelian unitary group on $V_i$. $\alpha$ is thus called the $U(1)$ gauge field on $V \times M$. In this section, we only consider the simpliest case, i.e.,

$$\alpha_i = A, \tag{47}$$

$$u(i) = u(j) \qquad (i, j = 1, 2, \cdots, n). \tag{48}$$

Here $A$ is a $U(1)$ gauge field on $V$. The physical meaning of the above asumptions is: there exist unique gauge field, i.e., the Maxwell electromagnetic field over all copies of $V$.

$\alpha$ can then be written by

$$\alpha = \sum_{i,j} \alpha_{ij} p_i d_f p_j + A. \tag{49}$$

As in Section 2, we introduce

$$a = \sum_{i,j} (1 + \alpha_{ij}) p_i d_f p_j. \tag{50}$$

Notice that

$$a_{ii} = 1. \tag{51}$$

One then has

$$\begin{aligned} a' &= u a u^*, \\ a'_{ij} &= u(i) a_{ij} u(j)^*. \end{aligned} \tag{52}$$

$A$ obeys the usual $U(1)$ gauge transformation rule,

$$A' = A + u d u^*. \tag{53}$$

The curvature of the connection $\alpha$ is given by

$$\Theta = d\alpha + \alpha^2. \tag{54}$$

It can be seen that $\Theta$ transforms in the usual way, $\Theta' = u \Theta u^*$. As a 2-form, $\Theta$ can be written as

$$\begin{aligned} \Theta &= d_s A + \sum_{ij} (d_s a_{ij}) p_i d_f p_j + \sum_{i,j,k} \theta_{ijk} p_i d_f p_j p_j d_f p_k, \\ \theta_{ijk} &= a_{ij} a_{jk} - a_{ik}. \end{aligned} \tag{55}$$

We see that $\Theta$ has a usual differential degree and a finit-difference degree $(\alpha, \beta)$ adding up to 2. Let us begin with the term in $\Theta$ of bi-degree $(2, 0)$:

$$\Theta^{(2,0)} = F = d_s A. \tag{56}$$

$F$ is the field strength over the continous space-time $V$.

Next, we look at the component $\Theta^{(1,1)}$ of bi-degree $(1, 1)$:

$$\Theta^{(1,1)} = \sum_{ij} (d_s a_{ij}) p_i d_f p_j. \tag{57}$$

$\Theta^{(1,1)}$ corresponds to the interaction between $V$ and $M$. It also obeys the field strength transformation rule:

$$\Theta'^{(1,1)} = u \Theta^{(1,1)} u^*. \tag{58}$$

Notice that there is a peculiar property to the discrete space-time: it is not $d_s + A$ but $d_s$ appears in (57)! The reason is: we take $\alpha_i = A$ at the begining. Otherwise, there will be a contribution from $\alpha_i$'s. This coincides with [2] (pp.561-576) and [5] (where we replace $U(2)$ by $U(1)$ and take $\omega_a = \omega_b$ in [2] and take $A = B$ in [5]).

Finally, we have the component $\Theta^{(0,2)}$ of degree $(0,2)$:

$$\Theta^{(0,2)} = \sum_{i,j,k} \theta_{ijk} p_i d_f p_j p_j d_f p_k. \tag{59}$$

$\Theta^{(0,2)}$ corresponds to the field strength over the finit space $M$.

Just as in Section 3, we use the formula (16) to deal with the finite-difference degrees, i.e.,

$$\pi(d_f p_i) = i[D, \pi(p_i)]. \tag{60}$$

We then obtain the action functinal over the discrete space-time $V \times M$:

$$S = \int_V \mathcal{L} d\nu. \tag{61}$$

The Lagrangian density is given by the following formulas:

$$\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_1 + \mathcal{L}_0, \tag{62}$$

$$\mathcal{L}_2 = \|\Theta^{(2,0)}\|^2 = |F|^2 = |d_s A|^2, \tag{63}$$

$$\mathcal{L}_1 = \|\Theta^{(1,1)}\|^2 = tr(d_s H)^2 = \frac{n}{n-1} \sum_i (d_s \phi)_i^2, \tag{64}$$

$$\begin{aligned} \mathcal{L}_0 &= trH^4 - 2\mu^2 trH^2 + n\mu^4 \\ &= \sum_{i,j,k,l} (\sum_\alpha e_i^\alpha e_j^\alpha e_k^\alpha e_l^\alpha) \phi_i \phi_j \phi_k \phi_l - \frac{2n}{n-1} \mu^2 (\sum_i \phi_i^2) + n\mu^4. \end{aligned} \tag{65}$$

**5. Geometric origin of phase transitions**

It is well know that the Landau-Ginzburg model is one of the most important models in the theory of phase transitions. It is a kind of 'metamodel' - it captures the essence of many models in this field.

The Hamiltonian density of the Landau-Ginzburg model is

$$\mathcal{H}_{LG} = \frac{1}{2}\alpha^2 |\nabla\phi|^2 + \frac{1}{2}\mu^2\phi^2 + \frac{1}{4!}\lambda(\phi^2)^2, \tag{66}$$

where $\phi$ is the order parameter, $\alpha, \mu^2, \lambda$ are phenomenological parameters. Notice that the temperature-dependent coefficient is $\mu^2$. This notation is not intended to imply that $\mu^2 > 0$.

Consider now the case when $V \times M = V \times \{2-\text{point space}\}$, it is easy to see from equations (64) and (65) that $\mathcal{L}_1$ and $\mathcal{L}_0$ correspond to the kinetic energy and the potential term of $\mathcal{H}_{LG}$ below the critical temperature respectively.

Moreover, in the continuous-spin formulation of the Ising model, see for example [21, 22], the effective Hamiltonian can be expressed as $\mathcal{H}_{LG}$ plus additional terms, and these additional terms are of no consequence if all one seeks is critical exponents. Notice that the Ising model is just the 2-state Potts model. One then can imagine that the effective Hamiltonian of the continuous-spin formulation of the $n$-state Potts model [19] may be related the action functional of $U(1)$ gauge field we derived in the previous section.

The effective Hamiltonian density of the $n$-state Potts model is

$$\begin{aligned} \mathcal{H}_P &= \sum_i (\nabla\phi_i)^2 + C_1(\sum_i \phi_i^2) + C_2 \sum_{i,j,k}(\sum_\alpha e_i^\alpha e_j^\alpha e_k^\alpha)\phi_i\phi_j\phi_k \\ &\quad + C_3 \sum_{i,j,k,l}(\sum_\alpha e_i^\alpha e_j^\alpha e_k^\alpha e_l^\alpha)\phi_i\phi_j\phi_k\phi_l + C_4(\sum_i \phi_i^2)^2, \end{aligned} \tag{67}$$

where $C_i (i = 1, 2, 3, 4)$ is a constant.

From equations (64) and (65), $\mathcal{L}_1$ really corresponds to the kinetic energy term of $\mathcal{H}_P$, $\mathcal{L}_0$ is contained in the potential term of $\mathcal{H}_P$. It might be asked: why there is such a correspondance? This is because that the set of states of the Potts model forms a $n$ point space, one then can build the gauge theory on $\{\text{continuous space}\} \times \{n\text{-point space}\}$.

In the Landau-Ginzburg model, the order parameter $\phi$ can be generalized to the vector order paramter $\mathbf{\Phi}$. Obviously, the action functional of $U(1)$ gauge theory over $\{\text{continuous space}\} \times \{n\text{-point space}\}$ is a generalized Landau-Ginzburg Hamiltonian below the critical temperature. This means that there may exist a nontrivial geometrical structure behind a class of phase transitions.

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